Related Topics
Rotation, angular velocity, torque, angular acceleration, angular moment, moment of inertia, rotational energy.

Principle
A known torque is applied to a body that can rotate about a fixed axis with minimal friction. Angle and angular velocity are measured over the time and the moment of inertia is determined. The torque is exerted by a string on a wheel of known radius with the force on the string resulting from the known force of a mass in the earth’s gravitational field. The known energy gain of the lowering mass is converted to rotational energy of the body under observation.

Tasks
1. Measure angular velocity and angle of rotation vs. time for a disc with constant torque applied to it for different values of torque generated with various forces on three different radii. Calculate the moment of inertia of the disc.
2. Measure angular velocity and angle of rotation vs. time and thus the moment of inertia for two discs and for a bar with masses mounted to it at different distances from the axis of rotation.
3. Calculate the rotational energy and the angular momentum of the disc over the time. Calculate the energy loss of the weight from the height loss over the time and compare these two energies.

Equipment

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<tr>
<th>Item</th>
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Fig.1: Experimental set up with turntable
Moment of inertia and angular acceleration

Set-up and procedure

1. Set the experiment up as seen in Fig.1. Connect the light barrier with counter according to Fig.2 to the cable release. Adjust the turntable to be horizontal – it must not start to move with an imbalance without other torque applied. Fix the silk thread (with the weight holder on one end) with the screw of the precision bearing or a piece of adhesive tape to the wheels with the grooves on the axis of rotation. Wind it several times around one of the wheels – enough turns, that the weight may reach the floor. Be sure the thread and the wheel of the precision pulley and the groove of the selected wheel are well aligned. Place the holding device with cable release in a way that it just holds the turntable on the sector mask and does not disturb the movement after release.

Note down the angle \( \varphi \) from the point of release to the point where the aperture plate enters the beam of the light barrier and the radius \( r_a \) of the selected wheel with groove. Measure the time \( t_1 \) necessary for the turntable to rotate about the angle \( \varphi \) by setting the mode selection switch of the light barrier to the symbol \( \text{"\( \varphi \)"} \). Press the reset button on the light barrier. Release the holder – the counter should stop, when the mask enters the light beam.

Measure the time necessary for the angle \( \varphi \) after more than one turn of the turntable by a shove with the hand. Nearly unaccelerated movement should be observable. Also measure the angular velocity after converting a defined amount of potential energy to rotational energy by allowing the weight \( m_a \) to pass only a given height before touching the ground.

2. Record measurements with two turntables mounted on the precision bearing with weight values used on the single turntable and with double the weight used on the single turntable for comparison.

Remove the turntables and mount the inertia rod to the precision bearing and the two weight holders symmetrically to the turntable for comparison.

You may also take a measurement with no weight \( m_a \). Start the turntable by a shove with the hand. Nearly unaccelerated movement should be observable. Also measure the angular velocity after converting a defined amount of potential energy to rotational energy by allowing the weight \( m_a \) to pass only a given height before touching the ground.

1. Take measurements with different accelerating weights \( m_a \) up to 100 g.

Note down measurement values with the thread running in different grooves, i.e. different radii \( r_a \) for the accelerating torque – adjust the position of the precision pulley to align thread and groove and wheel of the pulley (thread has to be horizontal).

Choose especially the weight on the end of the thread \( m_a \) such that the torque \( T = r_a \times F \) is constant – e.g. \( m_a = 60 \) g for \( r_a = 15 \) mm and \( m_a = 30 \) g for \( r_a = 30 \) mm and \( m_a = 20 \) g for \( r_a = 45 \) mm, each time the torque being \( T = m_a g r_a = 8.83 \) mNm with earth’s gravitational acceleration \( g = 9.81 \) N/kg = 9.81 m/s\(^2\). Also choose a weight with which you take a measurement for each groove radius.

You may also take a measurement with no weight \( m_a \). Start the turntable by a shove with the hand. Nearly unaccelerated movement should be observable. Also measure the angular velocity after converting a defined amount of potential energy to rotational energy by allowing the weight \( m_a \) to pass only a given height before touching the ground.

2. Record measurements with two turntables mounted on the precision bearing with weight values used on the single turntable and with double the weight used on the single turntable for comparison.

Remove the turntables and mount the inertia rod to the precision bearing and the two weight holders symmetrically to the rod with both the same distance to the axis \( r_2 \).

Take measurements with various masses \( m_i \) at constant \( r_2 \) and also with constant masses \( m_i \) at varied \( r_2 \) (both masses of course still mounted symmetrically) – accelerated with the same weight \( m_a \) (or with the same series of weights for high precision).

Theory and evaluation

The angular momentum \( \vec{r} \) of a single particle at place \( \vec{r} \) with velocity \( \vec{v} \), mass \( m \) and momentum \( \vec{p} = m \vec{v} \) is defined as

\[
\vec{r} = \vec{r} \times \vec{p}
\]

and the torque \( \vec{T} \) from the force \( \vec{F} \) is defined as

\[
\vec{T} = \vec{r} \times \vec{F},
\]

with torque and angular momentum depending on the origin of the reference frame. The change of \( \vec{T} \) in time is

\[
\frac{d}{dt} \vec{T} = \frac{d}{dt} (\vec{r} \times \vec{p}) = \frac{d}{dt} \vec{r} \times \vec{p} + \vec{r} \times \frac{d}{dt} \vec{p}
\]

and with

\[
\frac{d}{dt} \vec{r} \times \vec{p} = \vec{r} \times m \vec{v} = 0
\]

and Newton’s law

\[
\vec{T} = \frac{d\vec{p}}{dt}
\]

is the equation of motion becomes

\[
\frac{d}{dt} \vec{r} = \frac{d\vec{T}}{dt}
\] (1).
Moment of inertia and angular acceleration

For a system of N particles with center of mass $\vec{R}_{c.m.}$ and total linear momentum $\vec{P} = \sum m_i \vec{v}_i$, is

$$\vec{T} = \sum_{i=1}^{N} m_i (\vec{r}_i - \vec{R}_{c.m.}) \times \vec{v}_i + \sum m_i \vec{R}_{c.m.} \times \vec{v}_i = \vec{T}_{c.m.} + \vec{R}_{c.m.} \times \vec{P}.$$ \hspace{1cm} (1)

Now the movement of the center of mass is neglected, the origin set to the center of mass and a rigid body assumed with $\vec{r}_1 - \vec{r}_1$ fixed. The velocity of particle $i$ may be written as $\vec{v}_i = \vec{v} \times \vec{r}_i$ with vector of rotation

$$\vec{v} = \frac{d \vec{r}_i}{dt}$$ \hspace{1cm} (2)

constant throughout the body. Then

$$\vec{T} = \sum m_i \vec{r}_i \times \vec{v}_i = \sum m_i \vec{r}_i \times (\vec{v} \times \vec{r}_i).$$

With $\vec{d} = \vec{b} \times \vec{c}$ and $\vec{d} = \vec{b} (\vec{b} \cdot \vec{c}) - \vec{c} (\vec{b} \cdot \vec{b})$ is

$$\vec{T} = \sum m_i (\vec{d} \cdot \vec{r}_i - \vec{r}_i (\vec{d} \cdot \vec{r}_i))$$

with

$$\vec{r}_i \cdot \vec{d} = x_i \omega_x + y_i \omega_y + z_i \omega_z$$

and

$$I_x = \omega_x \sum m_i (r_i^2 - z_i^2) - \omega_z \sum m_i z_i x_i - \omega_y \sum m_i z_i y_i.$$ \hspace{1cm} (3)

The inertial coefficients or moments of inertia are defined as

$$I_{xx} = \sum m_i (r_i^2 - x_i^2)$$

$$I_{xy} = - \sum m_i x_i y_i$$

$$I_{xz} = - \sum m_i x_i z_i$$

and with the matrix $I = (I_{ij})$ it is

$$\vec{T} = \hat{I} \cdot \vec{\omega}$$ \hspace{1cm} (4)

and for the rotational acceleration $\vec{\omega} = \frac{d \vec{\omega}}{dt}$ is then

$$\vec{T} = \frac{d \vec{T}}{dt} = \hat{I} \cdot \frac{d \vec{\omega}}{dt} = \hat{I} \cdot \vec{\omega}.$$ \hspace{1cm} (5)

The rotational energy is

$$E = \frac{1}{2} \sum m_i \vec{v}_i^2 = \frac{1}{2} \sum m_i (\vec{\omega} \times \vec{r}_i)^2 = \frac{1}{2} I_{ij} \omega_i \omega_j$$

The coordinate axes can always be set to the “principal axes of inertia” so that none but the diagonal elements of the matrix $I_{xx} \neq 0$. In this experiment only a rotation about the z-axis can occur and $\vec{\omega} = \vec{\omega}_z$ with the unit vector $\hat{c}_z$. The energy is then

$$E = \frac{1}{2} I_{zz} \omega_z^2.$$ \hspace{1cm} (6)

The torque $T = m_a (g - \omega^2) r_a$ is nearly constant in time since the acceleration $a = \alpha \cdot r_a$ of the mass $m_a$ used for accelerating the rotation is small compared to the gravitational acceleration $g = 9.81 \text{ m/s}^2$ and the thread is always tangential to the wheel with radius $r_a$. So with (1), (2) and (3)

$$T = I_{zz} \frac{d^2 \varphi(t)}{dt^2} = m_a g r_a \varphi(t).$$ \hspace{1cm} (7)

The potential energy of the accelerating weight is

$$E = m_a g h(t) = -m_a g \varphi(t) \cdot r_a = \frac{1}{2} \frac{m_a g^2 r_a^2}{I_{zz}} \cdot t^2.$$ \hspace{1cm} (8)

For evaluation both the speed vs. angle ($\omega$ calculated by $t_0$) and the time vs. angle (using $t_1$) values may be used to determine $I_{zz}$. The time $t_1$ values are more precise in case the movement is still accelerated while the mask passes the light barrier.

Fig. 3 shows $t_1$ in dependance on torque $T$ for fixed $I_{zz}$, i.e. the moment of inertia of the turntable, and fixed $\varphi(t_1) = 215^\circ = 3.75 \text{ rad}$. The slope of the curve in the bilogarithmic plot is $-0.519$ compared to theoretical $-0.5$, since it follows from (8) that

$$t_1 = \sqrt{\frac{2 \varphi(t_1) I_{zz}}{T}}.$$
Fig. 3: Bilogarithmic plot of $t_1$ vs. $T$ for different torques.

Fig. 4 shows a bilogarithmic plot of angle vs. time with constant $I_{z,z}$ (of the turntable): With both values of torque the slope of the curve is nearly 2 as predicted by (8).

Fig. 5 shows a bilogarithmic plot of angular velocity vs. time with constant $I_{z,z}$ (of the turntable): With all three values of torque the slope of the curve is nearly 1 as predicted by (7).

Fig. 6 shows the $I_{z,z}$ values calculated from the measured $t_1$ values that the bar needed to reach $185^\circ = 3.23$ rad with a torque applied of $3.09$ mNm ($r_a = 15$ mm, $m_a = 21$ g, $g = 9.81$ m/s$^2$), according to (8):

$$I_{z,z} = \frac{T \cdot t_1^2}{2 \varphi}$$

- compared to the theoretical values, according to (3): $I_{z,z} = I_{rod} + m_i \cdot r_i^2$. $I_{rod} = 72$ kg cm$^2$. Two weights of 100 g each were mounted at $r_i$ from the axis of rotation – the plot shows a good linear dependence on $r_i^2$.

Fig. 7 shows measured $I_{z,z}$ values in dependence on the weight $m_i$ mounted to the bar at $r_i = 20$ cm in comparison to theoretical $I_{z,z} = I_{rod} + m_i \cdot r_i^2$. The linear dependence on $m_i$ can be seen.